Problemas T.F.

a)

$$f(x) = e^{-a|x|}$$

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-a|x|} e^{-i\omega x} dx = \int_{-\infty}^{0} e^{ax} e^{-i\omega x} dx + \int_{0}^{\infty} e^{-ax} e^{-i\omega x} dx$$

$$= \frac{e^{x(a-i\omega)}}{a-i\omega}\Big|_{-\infty}^{0} + \frac{e^{x(-a-i\omega)}}{-a-i\omega}\Big|_{0}^{\infty} = \frac{1}{a-i\omega} + \frac{1}{a+i\omega} = \frac{2a}{a^2+\omega^2}$$

Por fórmula de inversión: (~~por~~ $f \in L^1(\mathbb{R})$, f y f' continua x tramos)

$$\frac{f(x^-)+f(x^+)}{2} = \frac{1}{2\pi} \, VP \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega$$

Como f es continua en $\mathbb{R}$, $f(x) = \frac{f(x^-)+f(x^+)}{2}$.

Además, $\hat{f}(\omega)$ es absolutamente integrable $\Rightarrow VP\int \hat{f} e^{i\omega x} = \int \hat{f} e^{i\omega x}$

Luego:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2a}{a^2+\omega^2} e^{i\omega x} d\omega$$

$$\frac{\pi}{a} e^{-a|x|} = \underbrace{\int_{-\infty}^{\infty} \frac{1}{a^2+\omega^2} e^{i\omega x} d\omega}$$

definición de transformada de Fourier de $\frac{1}{a^2+\omega^2}$ en $(-x)$.

O sea: si $g(x) = \frac{1}{a^2+x^2} \Rightarrow$

$$\hat{g}(\omega) = \frac{\pi}{a} e^{-a|-\omega|} = \frac{\pi}{a} e^{-a|\omega|}.$$

b) $$\begin{cases} u'_x = u'_t & -\infty < x < \infty, \; t > 0 \\ u(x,0) = \dfrac{1}{x^2+1} & -\infty < x < \infty \end{cases}$$

Transformando: $\hat{U}(\omega,t) = \int_{-\infty}^{\infty} u(x,t) e^{-i\omega x} dx$

$$i\omega \hat{U}(\omega,t) = \hat{U}'_t(\omega,t)$$

La solución de esa EDO:

$$\hat{U}(\omega,t) = A(\omega) \cdot e^{i\omega t}$$

En $t=0$: $\hat{U}(\omega,0) = A(\omega) = \int u(x,0) e^{-i\omega x} dx = \int \frac{1}{1+x^2} e^{-i\omega x} dx$

$$A(\omega) = \pi e^{-|\omega|}$$

$$\hat{U}(\omega,t) = \pi e^{-|\omega|} e^{i\omega t}$$

$$u(x,t) = \mathcal{F}^{-1}\left(\pi e^{-|\omega|} e^{i\omega t}\right) = \frac{1}{1+(x+t)^2}$$

propiedad de traslación

a) Sean $f,g:\mathbb{R}\to\mathbb{R}$, a.i. y $g(t)=f(t-1)+f(-t-1)$
Mostrar que existe $\mathcal{F}(g)$ y calcularla en términos de $\mathcal{F}(f)$.

b) Calcular $\mathcal{F}^{-1}$ de las funciones:

i. $F(\omega)=\frac{1}{1+i\omega}$ ii. $F(\omega)=\frac{1}{(i\omega+1)^2+4}$.

a). $\hat{g}(\omega)=\int_{-\infty}^{\infty} g(t)e^{-i\omega t}\,dt = \int_{-\infty}^{\infty} f(t-1)e^{-i\omega t}\,dt+\int_{-\infty}^{\infty} f(-t-1)e^{-i\omega t}\,dt$

$$= \int_{-\infty}^{\infty} f(u)e^{-i\omega(u+1)}\,du + \int_{\infty}^{-\infty} f(z)e^{-i\omega(-z-1)}(-1)\,dz =$$

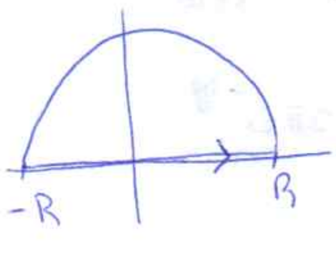
$$=e^{-i\omega}\int_{-\infty}^{\infty} f(u)e^{-i\omega u}\,du + \left(\int_{-\infty}^{\infty} f(z)e^{i\omega}\,dz\right)e^{i\omega}=$$

$$\boxed{= e^{-i\omega}\hat{f}(\omega)+e^{i\omega}\hat{f}(-\omega)}$$

(se puede obtener también $\hat{g}(\omega)$ usando propiedades)

b) i. $f(x)=\frac{1}{2\pi}VP\int_{-\infty}^{\infty} F(\omega)e^{i\omega x}\,d\omega = \frac{1}{2\pi}\lim_{R\to\infty}\int_{-R}^{R} \frac{1}{1+i\omega}e^{i\omega x}\,d\omega$

$$=\frac{1}{2\pi}\lim_{R\to\infty}\int_{-R}^{R} \frac{(1-i\omega)}{1+\omega^2}e^{i\omega x}\,d\omega$$



Sea $\tilde{f}(z)=\frac{(1-iz)}{(1+z^2)}e^{ixz}$

$$\int_C \tilde{f}(z)\,dz = 2\pi i \operatorname{Res}(f,i) = \cancel{2\pi i}\left(\frac{1-ii}{\cancel{2i}}\right)e^{ixi} = \pi\cdot e^{-x}\cdot 2$$

Sobre semicircunferencia C_R: como el integrando es producto de $g(z)=\frac{1-iz}{1+z^2}$ que verifica $|g(z)|\xrightarrow[|z|\to\infty]{}0$, por e^{ixz}, si $x>0$

Lema de Jordan asegura que $\lim\int_{C_R} g(z)e^{ixz}\,dz \xrightarrow{R\to\infty} 0$

Entonces: $\lim\limits_{R\to\infty}\int_{-R}^{R} \frac{1-i\omega}{1+\omega^2} e^{i\omega x}\, d\omega = \lim\limits_{R\to\infty} \int_{C} - \int_{C_R} =$

$$\lim_{R\to\infty}\int_{-R}^{R}\left(\frac{1-i\omega}{1+\omega^2}\right) e^{i\omega x}\, d\omega = 2\pi e^{-x} \quad \text{si } x>0$$

Luego: $f(x) = \frac{1}{2\pi} \lim\limits_{R\to\infty}\int_{-R}^{R} \frac{1-i\omega}{1+\omega^2} e^{i\omega x}\, d\omega = e^{-x}$ si $x>0$. $\circledast$

Si $x<0$? $\int \frac{1-i\omega}{1+\omega^2} e^{i\omega x}\, d\omega \underset{\substack{\downarrow \\ -x = y > 0}}{=} \int \frac{(1-i\omega)}{(1+\omega^2)} e^{-i\omega y}\, d\omega =$

$$= \int \frac{1-i\omega}{1+\omega^2}\left(\cos(\omega y) - i\,\text{sen}(\omega y)\right) d\omega = \int \frac{\cos(\omega y) - \omega\,\text{sen}(\omega y)}{1+\omega^2}\, d\omega +$$

$$i \int \underbrace{-\frac{\text{sen}(\omega y)}{1+\omega^2} - \frac{\omega\cos(\omega y)}{1+\omega^2}}_{=0 \text{ por integrando impar}} d\omega$$

Por otro lado: $\int_{-\infty}^{\infty} \frac{\cos(\omega y)}{1+\omega^2}\, d\omega = \lim\limits_{R\to\infty}\left(\int_C \frac{e^{i\omega y}}{1+\omega^2}\, d\omega - \overbrace{\int_{C_R} \frac{e^{i\omega y}}{1+\omega^2}\, d\omega}^{\to 0 \text{ porque } y>0} \right)$

$$= 2\pi i \cdot \frac{e^{i\cdot i y}}{2i} = \pi e^{-y}$$

Tenemos entonces: si $x<0$, $y=-x>0$

$$f(x) = \overbrace{\int \frac{\cos(\omega y)}{1+\omega^2}\, d\omega}^{A} - \overbrace{\int \frac{\omega\,\text{sen}(\omega y)}{1+\omega^2}\, d\omega}^{B} = \pi e^{-y} - \int \frac{\omega\,\text{sen}(\omega y)\, d\omega}{1+\omega^2}$$

Además, de $\circledast$ $\int \frac{\cos(\omega y)}{1+\omega^2}\, d\omega + \int \frac{\omega\,\text{sen}(\omega y)}{1+\omega^2}\, d\omega = 2\pi e^{-y}$

Tenemos: $\left.\begin{array}{l} A = \pi e^{-y} \\ A+B = 2\pi e^{-y} \end{array}\right\} \Rightarrow A - B = 0$

si $x<0$ $\therefore f(x) = 0$ $\longrightarrow$

Claro... es más sencillo probar que siendo

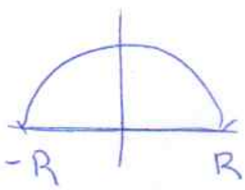
$$f(x) = \begin{cases} e^{-x} & \text{si } x>0 \\ 0 & \text{si } x<0 \end{cases} \Rightarrow \mathcal{F}(f)(\omega) = \frac{1}{1+i\omega},$$

calculando la transformada por definición.

ii) $$\mathcal{F}^{-1}(F(\omega))(x) = f(x) = \frac{1}{2\pi} \, VP \int_{-\infty}^{\infty} \frac{1}{(i\omega+1)^2+4} e^{i\omega x} \, d\omega \underset{i\omega+1 = i(\omega-i)}{=} \frac{1}{2\pi} \, VP \int_{-\infty}^{\infty} \frac{1}{4-(\omega-i)^2} e^{i\omega x} \, d\omega$$

Vamos a $\mathbb{C}$:

$$\tilde{g}(z) = \frac{1}{4-(z-i)^2} e^{izx}$$



$$\int_C \tilde{g}(z)\,dz = 2\pi i \left(\mathrm{Res}(\tilde{g}, 2+i) + \mathrm{Res}(\tilde{g}, -2+i) \right)$$

$$= 2\pi i \cdot \left(\frac{e^{i(2+i)x}}{-2 \cdot 2} + \frac{e^{i(-2+i)x}}{-2 \cdot (-2)} \right)$$

$$= \frac{\pi i}{-2} \left(e^{2ix} \cdot e^{-x} - e^{-2ix} \cdot e^{-x} \right)$$

$$= \frac{\pi i}{-2} e^{-x} \left(e^{2ix} - e^{-2ix} \right) = \pi e^{-x} \, \mathrm{sen}(2x)$$

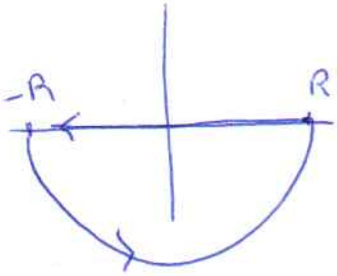
Sobre la semicircunferencia:

$$|\tilde{g}(z)| = \left| \frac{e^{izx}}{4-(z-i)^2} \right| \le \frac{e^{-x \, \mathrm{Im}\, z}}{|z-i|^2-4} \underset{\text{si } x>0}{\le} \frac{1}{|z-i|^2-4} \le \frac{1}{R^2-2R-5} = M_R$$

$$\left| \int_{C_R} \tilde{g}(z)\,dz \right| \le M_R \cdot \pi R \xrightarrow[R\to\infty]{} 0$$

Entonces: $$VP \int_{-\infty}^{\infty} \frac{1}{4-(\omega-i)^2} e^{i\omega x} \, d\omega = \int_C - \lim_{R\to\infty} \int_{C_R} \underset{\text{si } x>0}{=} \pi e^{-x} \, \mathrm{sen}(2x)$$

Si $x \leq 0$?



$\int_C \tilde{g}(z)\,dz = 0$ porque las singularidades quedan en exterior de C.

Sobre semicircunf $|z|=R$, $\Im m\, z<0$:

$$|\tilde{g}(z)| \leq \frac{e^{-x\,\Im m\, z}}{|z-i|^2-4} \leq \frac{1}{|z-i|^2-4} \leq \frac{1}{R^2-2R-5} = M_R$$

$\Im m\, z<0$, $x \leq 0$

$$\left|\int_{C_R} \tilde{g}(z)\,dz\right| \leq M_R \cdot \pi R \xrightarrow[R\to\infty]{} 0$$

Entonces:

$$VP\int_{-\infty}^{\infty} \frac{e^{i\omega x}}{4-(\omega-i)^2}\,d\omega = \int_C \tilde{g} - \lim_{R\to\infty}\int_{-R}^{R}\tilde{g} = 0$$

Así:

$$f(x) = \mathcal{F}^{-1}(F(\omega))(x) = \begin{cases} \frac{1}{2\pi}\pi e^{-x}\,\mathrm{sen}(2x) & \text{si } x>0 \\ 0 & \text{si } x\leq 0 \end{cases} = \begin{cases} \frac{e^{-x}\,\mathrm{sen}(2x)}{2} & \text{si } x>0 \\ 0 & \text{si } x\leq 0 \end{cases}$$

Como es dato $u(0,t)$ (y no $u'_x(0,t)$) usamos transformada seno (y no Transf. coseno).

$$\hat{U}(\omega,t) = \int_0^\infty u(x,t)\, \text{sen}(\omega x)\, dx$$

$$\int_0^\infty u'_t(x,t)\, \text{sen}(\omega x) dx \underset{\downarrow}{=} \frac{d}{dt}\int_0^\infty u(x,t)\, \text{sen}(\omega x) dx = \hat{U}'_t(\omega,t)$$

asumimos que se verifican hipótesis para hacer esto

$$\int_0^\infty u''_{xx}(x,t)\, \text{sen}(\omega x) dx = -\omega^2 \hat{U}(\omega,t) + \omega \underbrace{u(0,t)}_{\text{dato del problema } =0} = -\omega^2 \hat{U}(\omega,t)$$

La E.D. queda:

$$-\omega^2 \hat{U} = \hat{U}'_t \Rightarrow \hat{U}(\omega,t) = e^{-\omega^2 t} \cdot A(\omega)$$

Con $t=0$: $\hat{U}(\omega,0) = A(\omega)$

$$\hat{U}(\omega,0) = \int_0^\infty u(x,0)\, \text{sen}(\omega x) dx = \int_0^\infty [H(x) -$$

$$H(x) = \begin{cases} 1 & \text{si } x > 0 \\ 0 & \text{si } x \le 0 \end{cases} \qquad H(x-1) = \begin{cases} 1 & \text{si } x > 1 \\ 0 & \text{si } x \le 1 \end{cases}$$

$$H(x) - H(x-1) = \begin{cases} 1 & \text{si } 0 < x < 1 \\ 0 & \text{en otro caso} \end{cases}$$

Entonces: $A(\omega) = \int_0^1 \text{sen}(\omega x) dx = -\frac{\cos(\omega x)}{\omega}\Big|_0^1 = \frac{1-\cos(\omega)}{\omega}$

$$\hat{U}(\omega,t) = \frac{1-\cos(\omega)}{\omega} \cdot e^{-\omega^2 t}$$

$$u(x,t) = \frac{2}{\pi}\int_0^\infty \frac{1-\cos(\omega)}{\omega} e^{-\omega^2 t}\, \text{sen}(\omega x)\, d\omega$$